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Optimal Moving Grids for Time-Dependent Partial Differential Equations

A. J. WATHEN*

RIACS, NASA Ames Research Center, Moffett Field, California 94035

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Various adaptive moving grid techniques for the numerical solution of time-dependent partial differential equations have been proposed. The precise criterion for grid motion varies, but most techniques will attempt to give grids on which the solution of the partial differential equation can be well represented. We investigate moving grids on which the solutions of the linear heat conduction and viscous Burgers' equation in one space dimension are optimally approximated. Precisely, we report the results of numerical calculations of optimal moving grids for piecewise linear finite element approximation of PDE solutions in the least-squares norm. © 1992 Academic Press, Inc.

1. INTRODUCTION

For many time-dependent partial differential equations, particularly those which admit solutions with evolving sharp features such as fronts, use of moving grids can be an extremely useful tool for obtaining accurate numerical solutions efficiently. A number of moving grids procedures suggested for different problem classes have been [1, 12, 14, 7, 17, 8, 13].

In this communication we describe the results of a series of numerical experiments which are designed to examine how good a solution one can hope to achieve by continuously adapting the computational grid. The experimental setup is simple: piecewise linear finite elements (linear splines) on a closed interval in one dimension. The two boundary nodes are fixed, and the number of free internal nodes is not varied. Precisely, we investigate the problem of computing the optimal grid node positions of the best least-squares free knot linear spline approximation to the spatial solution of a PDE at different times. The loci of such optimal positions define an optimal trajectory for each node. The ensemble of these nodal trajectories defines the optimal moving grid in this norm. Piecewise linear representation is used here simply because it is a common choice. The choice of least-squares norm is more arbitrary; however, the results of Cullen and Morton [6] (see also [7, 2]) indicate that this may be an appropriate norm at least for the finite element method.

The problem of free knot spline approximation is nonlinear and great care is required to isolate the global minimum from potentially many locally optimal fits. In the work reported here, we have used a code based on the hybrid technique described by Loach and Wathen [10] for the approximation problem. Though no proof of global convergence exists for this algorithm, there is considerable evidence that the solutions it produces are indeed globally optimal. The testing of this routine used considerably more complicated test functions than the solutions of simple PDEs presented here.

We should emphasize that the computation of the optimal approximation is an expensive procedure and relies on exact knowledge of the PDE solution. The numerical techniques that we use here are therefore not at all suitable as practical moving grid methods for partial differential equations.

2. OPTIMAL GRID EXPERIMENT

It was desired to consider both parabolic and hyperbolic problems, thus two test problems were used.

The first was the parabolic problem of linear heat conduction

$$u_t = u_{xx}, \qquad 0 \leqslant x \leqslant 1, \qquad t \geqslant 0 \tag{2.1}$$

with homogeneous Neumann boundary conditions, $u_n = 0$ (here n is the normal direction to the boundary). In order that there be a rapidly computable analytic solution, initial data with few Fourier modes are employed. If a single mode is used, the PDE solution is simply an exponential decay of this original mode. The scale-invariance of the l_2 best fit equations then implies that the nodal positions in the

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optimal grid are fixed independently of time. The initial data are thus chosen as

$$u(x, t=0) = \cos \pi x + \cos 2\pi x \tag{2.2}$$

so that the analytic solution is simply

$$u(x, t) = e^{-\pi^2 t} \cos \pi x + e^{-4\pi^2 t} \cos 2\pi x. \tag{2.3}$$

For every instance in time, t, this describes the function to be approximated by the optimal free knot linear spline in space, x.

The second problem is the viscous Burgers equation

$$u_t + uu_x = \varepsilon u_{xx}, \qquad 0 \leqslant x \leqslant 1, \qquad t \geqslant 0 \tag{2.4}$$

which is nearly hyperbolic for small values of ε. A particular analytic solution of this problem can be found using the Cole–Hopf transformation giving

$$u(x, t) = f(\psi), \tag{2.5}$$

where

$$\psi = x - \mu t - \beta \tag{2.6}$$

and

$$f(\psi) = [\mu + \alpha + (\mu - \alpha)\, e^{(\alpha\psi/\varepsilon)}]/[1 + e^{(\alpha\psi/\varepsilon)}]. \tag{2.7}$$

ε is taken as 0.01, and the arbitrary constants α, β, and μ are chosen to be 0.4, 0.125, and 0.6, respectively, so that the solution is a right-ward moving steep front as t increases. The boundary conditions are time-dependent Dirichlet conditions taken from this exact solution.

For the near hyperbolic problem, the shock paths of the corresponding purely hyperbolic problem $u_t + uu_x = 0$ are likely to be near the optimal trajectories in this approximation experiment because of the advective nature of the solution. However, for the parabolic heat flow equation, no a priori "good" node trajectories are apparent.

The approximation problem is solved with fixed knots at the boundary points 0 and 1, but with free spline values at these end points. That is, if $x_0 = 0$, $x_{n+1} = 1$ are fixed knots and $x_1, x_2, ..., x_n$ are free knots, and a_i is the value taken by the linear spline S at x_i for $i = 0, 1, ..., n+1$, then the approximation problem is to minimise $\|f - S\|_{l_2[0,1]}$ with respect to x_i, $i = 1, 2, ..., n$, and a_i, $i = 0, 1, ..., n+1$, where f is the function to be approximated.

It is appropriate to comment that many practical moving/adaptive grid strategies are based on equidistribution of some monitor of the spatial error at each time step. That is, if $w(x)$ $(\geqslant 0)$ is a monitor function, then the equidistributing grid of n internal points $\{x_i\}$ can be computed to satisfy

$$\int_0^{x_i} w(x)\, dx = \frac{i}{n+1} \int_0^1 w(x)\, dx, \qquad i = 1, ..., n. \tag{2.8}$$

The results of Carey and Dinh [4] show that the free knot linear spline interpolation problem in the least-squares norm is approximately equivalent to equidistribution of the monitor function $(f'')^{(2/5)}$.

Coyle, Flaherty, and Ludwig [5] analyse dynamic equidistribution using a time-dependent monitor $w(x, t)$:

$$\int_0^{x_i} w(x, t)\, dx = \frac{i}{n+1} \int_0^1 w(x, t)\, dx = c(t), \qquad i = 1, ..., n. \tag{2.9}$$

Equation (2.9) can be solved for the equidistributing grid using, for example, Newton iteration at each time step. Alternatively, differentiation of (2.9) with respect to time gives ordinary differential equations for the grid node motion. For dissipative parabolic systems, Coyle *et al.* show that for most choices of w, these ordinary differential equations are unstable to small perturbations of the initial node positions. Stability conditions on $w(x, t)$ are given and a more stable, approximately equidistributing dynamic strategy which involves a "relaxation" parameter, λ, is proposed. For large values of λ, the grid is more accurately equidistributing, but the ordinary differential equations are stiff. For small values, one obtains less accuracy and less stiffness.

3. COMPUTATIONAL RESULTS

We present results for a typical run on each of the test problems. For the linear heat flow problem with 10 internal nodes Fig. 1 shows the loci of the optimal nodal trajectories, the analytic solution at selected times, and the best free knot spline approximants to the analytic solution at these instants, all plotted in the u–x plane. That is, we have plotted the analytic solution (dotted line) and its best least-squares fit (dashed line) at a number of discrete times, and at many smaller time intervals we have plotted the optimal location in u–x space of the optimal node position. These latter are joined by the solid line to indicate the loci of the node points which define the optimal grid.

One immediate observation is that the optimal node point trajectories undergo a discontinuous jump at two particular instances, $t \simeq 0.007$ and $t \simeq 0.034$ during $0 \leqslant t \leqslant 0.091$, which is the time interval chosen for the experiment. Figure 2 shows only the nodal trajectories for the same problem in the x–t plane. The least-squares error of the optimal linear spline was found to be continuous

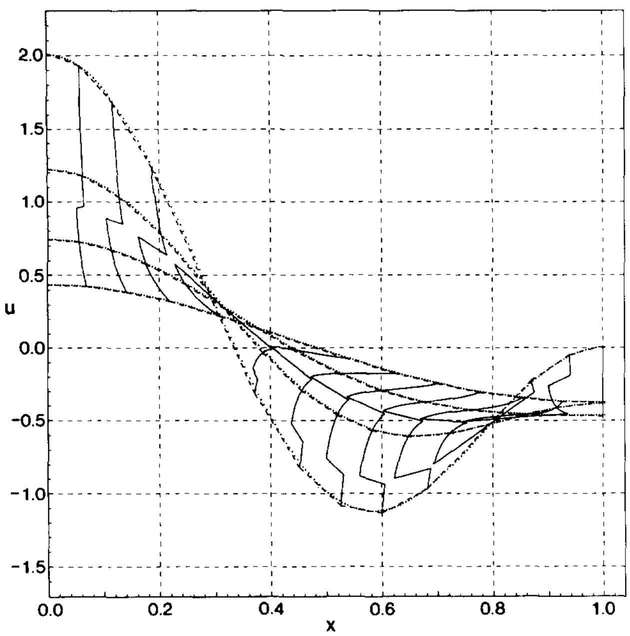

FIG. 1. Linear heat flow equation with 10 free nodes: exact solution (dotted line), best fits (dashed line), and optimal nodal trajectories (full line) all in u–x space.

as a function of time as expected, but apparently had derivative discontinuities at the two instances when the node positions jumped.

Several other experiments were performed on this heat flow problem with qualitatively similar results. We comment that there appear to be no "boundary effects" caused by our choice of end conditions for this test problem.

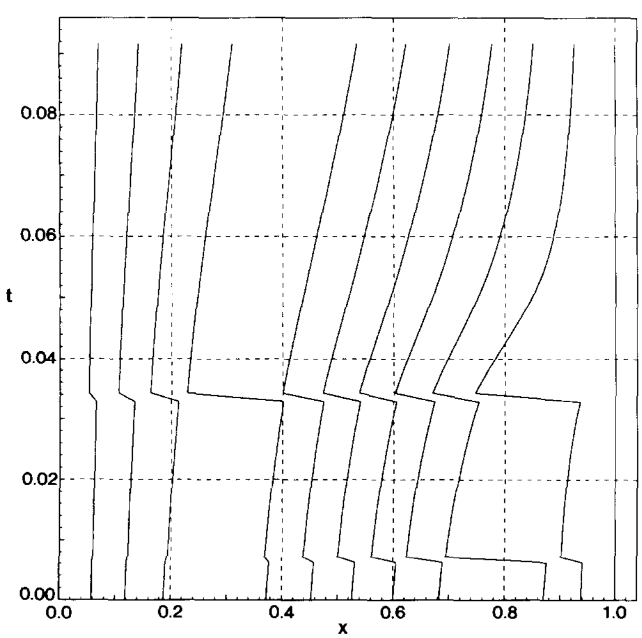

FIG. 2. Linear heat flow equation with 10 free nodes: optimal nodal trajectories in x–t space.

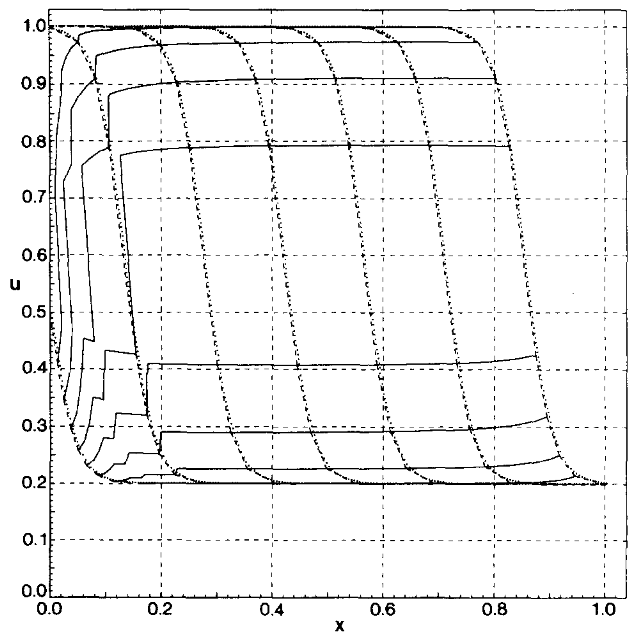

FIG. 3. Viscous Burgers equation with eight free nodes: exact solution (dotted line), best fits (dashed line), and optimal nodal trajectories (full line) all in u–x space.

Several experiments for the viscous Burgers equation were performed. We present only the results for eight free nodes. The graphical results are presented in the same way as for the heat flow equation in Fig. 3.

This experiment was run from $t = -0.23$, when the steep front is just moving into the domain, until $t = 1.21$. The results show the discontinuous movement of the nodes as the front enters the domain followed by smooth wave-like motion as the front moves across the domain. The l_2 error is again continuous. We see in this example the effect of the solution entering through the domain boundary. Even for the corresponding hyperbolic inviscid Burgers equation, a boundary condition at this left-hand boundary would be required as the characteristics enter the domain here.

4. CONCLUSIONS

By computing the moving grids on which the analytic solution of two simple partial differential equations can be best represented at every instant of time by a linear spline with a fixed number of nodes, we find that the optimal moving grids have discontinuous nodal trajectories. For more complicated time-dependent partial differential equation problems, we must expect to obtain similar behaviour in general.

We conclude that global optimal approximation continuously in time is not a useful property to try to achieve with a moving grid strategy because of the difficulties in trying to time integrate discontinuous optimal nodal paths.

Our computations lead us to believe that there are likely to be smooth locally optimal paths for the problems examined and that the discontinuities appear in the globally optimal trajectories due to exchange of global optimality between local minima. This would suggest two possible remedies.

Periodic regridding would allow computation to proceed along smooth nodal trajectories with occasional discontinuities. Selecting appropriate times to regrid may not be straightforward.

Regularisation may be used to effect a smooth transient from one grid to another—we might expect stiffness in the time integration if nodal paths were to adjust on a much more rapid time scale than the evolution of the partial differential equation solution itself. This appears to be bourne out in the work of Coyle *et al.* [5], where the accuracy/stiffness trade-off is controlled by a relaxation parameter.

It may be possible to follow locally optimal grid paths which are smooth—this could be adequate for some problems. For other problems, there must be expected to be locally optimal grids on which the error is large. These results and observations are of relevance to adaptive gridding strategies in the method of lines [11, 3, 16, 15, 9].

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